For all questions, answer choice (E) NOTA means that none of the given answers is correct. All inverse trigonometric functions obey traditional ranges, and $\lfloor x \rfloor$ signifies the greatest integer less than or equal to x. Good Luck!

1. In the Weierstrass Substitution, $u = \tan(x/2)$. If this relationship is true and $0 < x < \pi/2$, find $\sin(x) - \frac{dx}{du} + \cos(x)$ in terms of u.

   (A) $\dfrac{u-1}{u^2+1}$ (B) $\dfrac{u+1}{u^2+1}$ (C) $\dfrac{(u-1)^2}{u^2+1}$ (D) $\dfrac{(u+1)^2}{u^2+1}$ (E) NOTA

2. Akash defines A_n as the total area bounded by $y = x^{4n+1}$ and $y = x^{4n+3}$. Find the sum of A_n across all non-negative integer values of n.

   (A) $\pi/4$ (B) $\pi/2$ (C) $\ln\sqrt{2}$ (D) $\ln 2$ (E) NOTA

3. Tanusri and Tanvi still love tangents! They are both on the graph of $y = |\tan(x)|$, but Tanusri's x-coordinate is $-a$ while Tanvi's x-coordinate is a, where $0 < a < \pi/2$. The tangent lines to $y = |\tan(x)|$ at Tanusri and Tanvi's positions intersect at a point with y-coordinate $-a$. What is the area of the triangle with vertices at Tanusri, Tanvi, and the origin?

   (A) 1/4 (B) 1/2 (C) 1 (D) 2 (E) NOTA

4. If $f(x) = 5\sqrt{2 + 2\cos(x)}$, evaluate the sum $S = f(\frac{\pi}{3}) + f'(\frac{\pi}{3}) + f''(\frac{\pi}{3}) + \ldots$

   (A) $2\sqrt{3} + 4$ (B) $4\sqrt{3} - 2$ (C) $8\sqrt{3} - 2$ (D) $\sqrt{3} + 2$ (E) NOTA

5. Let a_m be the value of a greater than 1 for which the graphs of $y = x^a$ and $y = a^x$ intersect at as few first-quadrant points as possible. Find the sum of the digits of $\lfloor 10a_m \rfloor$.

   (A) 3 (B) 5 (C) 9 (D) More than one possible answer (E) NOTA

6. Evaluate

$$\int_0^{3\pi/2} (\tan(\arctan(x)) - \arctan(\tan(x)))dx$$

   .

   (A) $\frac{\pi^2}{4}$ (B) $\frac{3\pi^2}{4}$ (C) $\frac{3\pi^2}{2}$ (D) $\frac{9\pi^2}{4}$ (E) NOTA

7. Navya randomly chooses a point along the curve $y = \sqrt{3-x}$ within the first quadrant and draws the line normal to the curve at that point. What is the minimum possible y-intercept of this normal line?

   (A) $\frac{-3\sqrt{7}}{4}$ (B) $\frac{-5\sqrt{30}}{9}$ (C) $\frac{-7\sqrt{5}}{8}$ (D) $\frac{-5\sqrt{6}}{3}$ (E) NOTA

8. Rohan gives Karthik the seemingly simple math problem of finding the value of c which satisfies the Mean Value Theorem for $y = ax^n$ on $[0, b]$ $(a, b > 0)$, but clever Karthik disputes the question because Rohan did not state whether he was asking about MVT for integrals or for derivatives. If c_I is the answer to the question for MVT of integrals, and c_D is the answer for MVT of derivatives, find $\lim\limits_{n\to\infty} c_I/c_D$.

   (A) 0 (B) 1 (C) e (D) Non-finite (E) NOTA

9. While playing Fruit Ninja, Shrung slices a 2D watermelon defined by the curve $x^2 - 10x + y^2 - 12y + 57 = 0$ along the line $y = 16 - 2x$, forming two distinct regions of finite non-zero area. The one below the line has a center of mass (x_1, y_1) while the one above the line has a center of mass (x_2, y_2). Find $x_1 + y_1 + x_2 + y_2 + 6\pi\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

   (A) 22 (B) 33 (C) 38 (D) 54 (E) NOTA

10. Find the maximum value of x on the graph of the following curve:

$$y = e^{xe^{xe^{xe^{\cdots}}}}$$

(A) $\frac{1}{e}$ (B) 1 (C) e (D) There is no maximum (E) NOTA

11. Let R be the region bound by the y-axis and the curve from the previous question. Find the total volume of the figure formed when R is revolved about the y-axis.

(A) π (B) 2π (C) $e\pi$ (D) Volume is non-finite (E) NOTA

12. Evaluate the following integral, where $\{x\}$ is defined as the "decimal" part of x (for instance, $\{5.45\} = 0.45$, $\{0.8\} = 0.8$, and $\{6\} = 0$):

$$\int_0^{20} 15x\sqrt{\{x\}}\, dx$$

(A) 1900 (B) 1920 (C) 2000 (D) 2020 (E) NOTA

13. To assess her astounding abilities, Anjali attempts to awkwardly approximate the root of $f(x) = x^{2021}$ by performing 2021 iterations of Newton's method with an initial guess of $x_0 = 136$ (her favorite number). Find the x-value she obtains after this entire process, to the nearest integer.

(A) 1 (B) 10 (C) 50 (D) 100 (E) NOTA

14. An nth antiderivative of $f(x)$ is a function whose nth derivative is $f(x)$. If $g(x)$ is one of the 2021-st antiderivatives of $f(x) = x$, and 1 is a root of the 2020-th derivative of $g(x)$, what is the sum of the squares of the roots of $g(x)$? (Hint: $2021^2 = 4084441$)

(A) 4082420 (B) 4082421 (C) 4086461 (D) 4086462 (E) NOTA

15. Shreyas made the mistake of trying to use Wolfram Alpha's experimental speech-to-text feature. Help him out by evaluating the following limit:

$$\lim_{x\to\infty} x(4 + 4/x + 1/x^2)\sqrt[x]{2 + 1/x} - 4x$$

(A) 0 (B) 4 (C) $4 + 2\ln 2$ (D) $4 + 4\ln 2$ (E) NOTA

16. While studying for a microeconomics test, Dylan learns that a market can be represented by a downward-sloping "demand curve," which relates the quantity demanded by consumers (Q) on the horizontal axis to the price of a good (P) on the vertical axis. The price elasticity of a demand curve is the ratio of an infinitesimal percent change in Q to the corresponding infinitesimal percent change in P - for instance, if a 0.05% increase in price caused a 0.01% decrease in quantity demanded, the price elasticity at that point would be *approximately* -0.2.

Using the above information, find the exact price elasticity of demand for the demand curve $P = 5 - 2Q^2$ when $Q = 1$.

(A) -0.75 (B) -1.5 (C) -3 (D) -4 (E) NOTA

17. *(This question uses the information from the previous question)* The market for sweet tea is "double elastic," meaning its price elasticity at all quantities is -2. The demand curve is also continuous and differentiable for all positive values of P and Q. If consumers buy 20 gallons of sweet tea when the price is \$15.00, find the number of gallons that will be purchased when the price is \$10.00.

    (A) 25 (B) 30 (C) 40 (D) 45 (E) NOTA

18. In an effort to escape Nitish, Eric starts at $(x_0, 21\sin(x_0))$ and marches steadily to the right along the graph of $y = 21\sin(x)$ while Nitish sits at the origin. Nitish notices that the distance between them is always increasing as Eric walks in this zigzag pattern. To the nearest multiple of 10, what is the least possible value of x_0?

    (A) 210 (B) 220 (C) 420 (D) 440 (E) NOTA

19. Vishnav is at the vertex of a swimming pool in the shape of an equilateral triangle with side length 4 m, and is trying to get to his car, the midpoint of the opposite side, as quickly as possible. He can run along the edge of the pool at $\sqrt{2}$ m/s and swim at 1 m/s, but once he swims he is wet and can no longer run. What is the minimum amount of time (in seconds) it will take him to reach his car?

    (A) $\dfrac{\sqrt{6}+3\sqrt{2}}{2}$ (B) $\dfrac{2\sqrt{3}+\sqrt{6}}{3}$ (C) $\dfrac{3\sqrt{2}-\sqrt{6}}{2}$ (D) $\dfrac{3\sqrt{6}+\sqrt{2}}{2}$ (E) NOTA

20. Find the first-quadrant area bound by $y = \arctan(\frac{1}{x})$, $y = \dfrac{x}{1+x^2}$, and the y-axis. (Hint: $\arctan(\frac{1}{x}) > \frac{x}{1+x^2}$ for all positive values of x).

    (A) $\frac{1}{2}$ (B) 1 (C) $\frac{\pi}{2}$ (D) Area is non-finite (E) NOTA

21. Evaluate the following infinite series:

$$\frac{1}{3(1!)} + \frac{1}{4(2!)} + \frac{1}{5(3!)} + \frac{1}{6(4!)} \cdots$$

    (A) $e-2$ (B) $2e-5$ (C) $\frac{2e^2-13}{2}$ (D) $\frac{e-1}{2}$ (E) NOTA

22. For their MAO Banquet, Ananya and Akhil are splitting a cake and want equal amounts. The cake is in the shape of the region bounded by the coordinate axes and $y = 4 - x^2$ in the first quadrant. To maintain the parabolic theme, Dylan cuts the cake along another parabolic curve with vertex at the cake's right angle. If Dylan's cut indeed results in equal areas on either side, and the length of the cut is L, find $12L$.

    (A) $6\sqrt{37} + \ln(6+\sqrt{37})$

    (B) $9\sqrt{37} - \ln(3+\sqrt{37})$

    (C) $3\sqrt{41} + \ln(6+\sqrt{37})$

    (D) $9\sqrt{37} + \ln(3+\sqrt{37})$

    (E) NOTA

23. Of course, in real life cakes are rarely 2-dimensional. Dylan, Farzan, Prabhas, and Mihir each want to make a 3-D cake by revolving the first-quadrant region bounded by the coordinate axes and $y = 4 - x^2$ around a certain axis. Let:

    $D =$ the volume when Dylan revolves the cake around $x = 0$.
    $P =$ the volume when Prabhas revolves the cake around $y = 0$.

$M =$ the volume when Mihir revolves the cake around $x = 4$.
$F =$ the volume when Prabhas revolves the cake around $y = 5 - x$.

Rank D, P, M, and F from least to greatest to find which cake is superior!

(A) $D < P < M < F$
(B) $P < M < D < F$
(C) $D < P < F < M$
(D) $P < D < F < M$
(E) NOTA

24. For Vishal's birthday, he decides to form a 3-D cake by rotating the 2-D cake (first-quadrant region bounded by $y = 4 - x^2$ and the coordinate axes) about the y-axis. Then, he takes a large bite in the shape of a rectangular prism with one face on the circular base of the 3-D cake. If Vishal takes the largest bite possible (in terms of volume), what proportion of the cake's original volume does he consume?

(A) $\frac{1}{2\pi}$ (B) $\frac{1}{\pi}$ (C) $\frac{1}{2}$ (D) $\frac{2}{\pi}$ (E) NOTA

25. Prabhas the Probability Master starts at a random position in a square of side length 1 meter, then moves a distance of 1 meter in a completely random direction (within the plane of the square). Find the probability that Prabhas is no longer within the square at the end of his movement.

(A) $\frac{3}{\pi}$ (B) $\frac{3}{2\pi}$ (C) $\frac{2}{\pi}$ (D) $\frac{8}{3\pi}$ (E) NOTA

26. A particle travels in the polar plane with $r(t)$ and $\theta(t)$ differentiable with respect to t. Relpek notes that if $A(t_1, t_2)$ is the area bound by the line $\theta = \theta(t_1)$, the line $\theta = \theta(t_2)$, and the path of the particle from $t = t_1$ to $t = t_2$, then $A(a, a + \tau) = A(b, b + \tau)$ for all values of a, b, and τ. Given that $r(t) = (\theta(t))^2$, $\theta(0) = \pi$, and $r'(0) = -2\pi^2$, find the least positive value of t at which the particle crosses the line $\theta = \frac{\pi}{2}$.

(A) $\dfrac{3}{16}$ (B) $\dfrac{16}{85}$ (C) $\dfrac{31}{160}$ (D) $\dfrac{15}{64}$ (E) NOTA

27. Kevin wishes to create the perfect integral integral. Help him out by finding the largest integral value of a for which the following integral evaluates to an integer:

$$\int_a^\infty \frac{(\ln 1000)^{1001}(x-a)^{1000}}{1000^x}dx$$

(A) 82 (B) 83 (C) 84 (D) 85 (E) NOTA

28. Tanmay is tied to a fixed point on the circumference of a pen in the shape of a circle with radius 1 m by a leash of length π m. Tanmay can graze freely in the grassy field outside the pen, but cannot enter the pen. Find the total area of grass, in square meters, that Tanmay can graze. (Hint: Polar integration about the instantaneous contact point.)

(A) $\frac{1}{2}\pi^3$ (B) $\frac{2}{3}\pi^3$ (C) $\frac{5}{6}\pi^3$ (D) $\frac{8}{9}\pi^3$ (E) NOTA

29. Estimate the value of the following integral to the nearest integer:

$$\int_0^{\ln 2} e^{e^x}\,dx$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

30. Hooray! End of the test! Using the Weierstrass substitution, make quick work of this pesky final integral:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\csc(x)}$$

(A) $\frac{\pi}{2}-1$ (B) $2-\frac{\pi}{2}$ (C) $2\ln 2-\frac{\pi}{4}$ (D) $1-\frac{\pi}{4}$ (E) NOTA